$\underbrace{\text{Hyper}}\underbrace{\text{kähler}}$ manifolds

Def. (M, g) AG

I, J, K – complex str. DG

$I : TM \to TM$, $I^2 = -Id$

$IJ = -JI = K$

$d\omega_I = d\omega_J = d\omega_K = 0$

$\omega_I(X, Y) := g(X, IY)$

then M is HK

C. Voisin

Hodge ~~theory~~ and complex geom

Rmk

x

$x + iIx$ $\;-i$

$x - iIx$ $\;i$

$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$

Berger Th: oriented AG

Kähler

$\cancel{CY} \supset HK$ ~ Hyperkähler

Calabi Yau → CY

$su(n)$ $\quad sp(n)$

Spin(7)

G_2

Def (M, I) is called holomorphically symplectic if $\exists$ non-dg. $(2,0)$ form Ω

In particular, HK $\Rightarrow$ hol. symp)

(M, I) $\quad \Omega = \omega_J + i\omega_K$

Def IHS = hyperkähler, irreducible

(kähler hol. symp) + kähler (CY theorem)

simply-connected,

$H^{2,0} = \mathbb{C}\langle\Omega\rangle$

Cohomology :

$H^{p,q}$

$h^{p,q} = h^{q,p}$

$h^{n-p,n-q} = h^{p,q}$

$h^{n-p,q} = h^{p,q}$

example

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\boxed{K3}$

Remark : $\dim_{\mathbb{C}} X = 2$, $\quad kod = 0$

$(K_X = 0, \; K_X^2 = 0)$

K3, T, Enriques, hyperelliptic, Kodaira

IHS $\quad$ = K3/G $\quad$ E×E/G $\quad$ (~Thimm...)

non-kähler

hol. symple

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ 1 & 2 & 1 \\ 2 & & 1 \\ & 1 & \end{array}$$

HK: examples

classification

Beauville–Bogomolov decomposition.

Conj (Beauville)
A number of HK manifolds is finite in every dimension

$M = CY_i \times \underbrace{IHS_j}_{} \times T$

$\uparrow$

CY

Examples:
- $Hilb_n(K3)$ $\quad Hilb_2(K3) \downarrow Sym^2(K3)$
- $Hilb_n(T) \xrightarrow{\Sigma} T \ni 0 \quad \Sigma^{-1}(0) = \boxed{K_{n-1}(T)}$

$T^n \xrightarrow{t_1 + \dots + t_n} T$

$n=2$ $\quad Hilb_2(T) \to T$

$K_1(T)$ = Kummer K3 surf.

gen. Kummer

- OG6, OG10

"moduli spaces of semistable sheaves of rank 2"

on K3, on T with some conditions.

OG10, OG6 (c_1, c_2)

Fano variety of lines on cubic surface

$\cong$

$Hilb_2(K3)$

non-Kähler "IHS"?

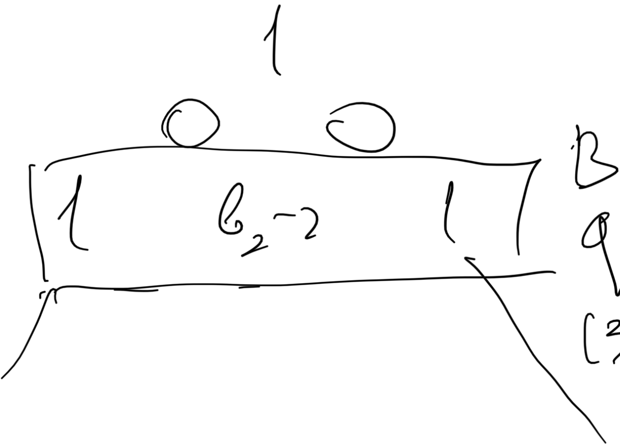
(non-Kähler simply-connected hol. sympl. with $H^{2,0}=\mathbb{C}\langle\sigma\rangle$)

Guan–Bogomolov: there is such example

($Q^2 \to \mathrm{Hilb}_n(\mathrm{Kod})$)

Beauville conj.?

Torelli th. $(3,19)$-K3



BBF q $(3b_2-3)$

True for HK: Verbitsky, Bakker–Lehn, Huybrechts

small Beauville conj.:

b_2 is bounded in any dimension

$\dim_{\mathbb{C}}=4$ $\quad b_2 \le 23$

Laza–Robles–Kim–Green

they found a condition when b_2 is bounded

$SO(\ell_1, \ell_2 - 2)$ B_4, D_4

($\frown$, Saran) ;

$\boxed{\ell_2(4)}$